A Short Note on D=3 N=1 Supergravity

Sunny Guha

December 13, 2015

1 Why 3-dimensional gravity ?

Three-dimensional field theories have a number of unique features, the massless states do not carry helicity, so that the associated degrees of freedom can generally be described by scalar fields. Pure gravity and supergravity are topological theories and do not give rise to physical (i.e. propagating) degrees of freedom. Apart from conical singularities at the location of matter sources, space-time is flat. A further motivation for studying three-dimensional supergravity is the important role it plays in the construction of two-dimensional supergravity theories via dimensional reduction.

In 2+1 dimensions, the Weyl tensor vanishes identically, and the full curvature tensor is determined algebraically by the curvature scalar and the Ricci tensor:

$$R_{\mu\nu\rho\sigma} = g_\mu \rho R_{\nu\sigma} + g_{\nu\sigma} R_{\mu\sigma} - g_{\nu\rho} R_{\mu\sigma} - g_{\mu\sigma} R_{\nu\rho} - \frac{1}{2}(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) R \quad (1)$$

In particular, this implies that any particular solution of the vacuum Einstein field equations is flat, and that any solution of the field equatiuons wirth a cosmological constant has constant curvature.

$$R_{\mu\nu} = 2\Lambda g_{\mu\nu} \quad (2)$$

Physically a 3-dimensiaonal spacetime has local degrees of freedom, ie there are no gravitational waves in the classical theory and no gravitons in the quantum theory.

2 Degrees of Freedom

3d N=1 supergravity, for On-shell has the multiplet $\{e^a_\mu, \psi^\alpha_\mu\}$, with no degrees of freedom. Since on shell degree of freedom for gravitino and graviton are,

$$(d-1)(d-2)/2 - 1 = 0 \qquad \text{Graviton} \qquad (d-3)2^{[d/2]}/2 = 0 \qquad \text{Gravitino} \quad (3)$$

However for the offshell case , we have the following degrees of freedom,

$$(d)(d-1)/2 = 3 \qquad \text{Graviton} \qquad (d-1)2^{[d/2]} = 4 \qquad \text{Gravitino} \tag{4}$$

Since the degrees of freedom do not match, for off-shell we need to add another bosonic degree of freedom in the form of an auxiliary field S. Thus the off-shell multiplet is $\{e^a_\mu, \psi^\alpha_\mu, S\}$.

3 Gamma Identites

We will take our 3d metric to be mostly plus,

$$\eta_{\mu\nu} = diag(-1, 1, 1) \tag{5}$$

We will use (m,n,..) for flat space indices and $(\mu, \nu, .)$ for curved space indices. For the three dimensional gamma matrices we choose the following representation,

$$(\gamma^0)^\alpha_\beta = -i\sigma^2 \qquad (\gamma^1)^\alpha_\beta = \sigma^1 \qquad (\gamma^2)^\alpha_\beta = \sigma^3 \tag{6}$$

Upper and Lower Gamma matrices are given by,

$$\gamma^\mu = \{-1, -\sigma^3, \sigma^1\} \tag{7}$$

$$\gamma_\mu = \{1, \sigma^3, -\sigma^1\} \tag{8}$$

In 3 dimensions the gamma matrices are particularly simple and take the form of levi civita,

$$\gamma^{mnp} = -\epsilon^{mnp} \tag{9}$$

with curved indices ϵ becomes a density.

$$e\gamma^{\mu\nu\rho} = -\epsilon^{\mu\nu\rho} \tag{10}$$

$$\epsilon^{\mu\nu\rho}\gamma_\rho = -\gamma^{[\mu}\gamma^{\nu]} \tag{11}$$

Contraction between ϵ of different indices is given as,

$$\epsilon^{\mu\nu\rho}\epsilon_{mnr} = -6ee^{[\mu}_m e^\nu_n e^{\rho]}_r \tag{12}$$

$$\gamma_{mn} = -\epsilon_{mnr}\gamma^r \tag{13}$$

We would also need the use of following identities

$$\epsilon^{\mu\nu\rho}\gamma_\rho = -\gamma^{[\mu}\gamma^{\nu]} \tag{14}$$

$$\gamma^\mu\gamma^\rho\gamma^\sigma = \gamma^{[\mu}\gamma^\rho\gamma^{\sigma]} + \eta^{\mu\rho}\gamma^\sigma + \eta^{\rho\sigma}\gamma^\mu - \eta^{\mu\sigma}\gamma^\rho \tag{15}$$

4 The Action

The Full "pure supergravity" action(on-shell) for D=3 N=1 Supergravity is given by,

$$S = \frac{-1}{8k^2}\int d^3xeR - \frac{1}{2}\int d^3xe\bar{\Psi}\gamma^{\mu\nu\rho}D_\nu(\omega)\Psi_\rho \tag{16}$$

The Ricci Scalar written in terms of veilbein indices is,

$$R = R_{\mu\nu}^{\ \ mn}(\omega)e_m^\nu e_n^\mu \tag{17}$$

The Riemann tensor in terms of spin connection is given as,

$$R_{\mu\nu}^{\ \ mn}(\omega) = \partial_\mu\omega_\nu^{mn} - \partial_\nu\omega_\mu^{mn} + \omega_\mu^{mp}\omega_\nu^{pn} - \omega_\nu^{mp}\omega_\mu^{pn} \tag{18}$$

The covariant derivative is defined as ,

$$D_{nu}\Psi_\rho = \partial_\nu\Psi_\rho + \frac{1}{4}\omega_{nu}^{mn}\gamma_{mn}\Psi_\rho \tag{19}$$

In the action κ^2 is the gravitational constant with mass dimensions -1 in 3 dimensions. Also $e = det(e_\mu^m) = \sqrt{-g}$.
The supersymmetric transformation for the graviton and gravitino field are,

$$\delta e_\mu^m = 2k\bar{\epsilon}\gamma^m\Psi_\mu \tag{20}$$

$$\delta\Psi_\mu = \frac{1}{\kappa}D_\mu(\omega)\epsilon = \frac{1}{\kappa}(\partial_\mu\epsilon + \frac{1}{4}\omega_\mu^{mn}\gamma_m\gamma_n\epsilon) \tag{21}$$

We will use these to check the supersymmetry invariance of the action.

5 SUSY Algebra

We will be using the 1.5 formulation. Using equations (9) the action of Gravitino can be written as,

$$I_2 = \int d^3x\frac{1}{2}\epsilon^{\mu\rho\sigma}\bar{\Psi}_\mu D_\rho(\omega)\Psi_\sigma \tag{22}$$

SUSY variation of this action gives us ,

$$\delta I_{3/2} = \frac{1}{\kappa}\int d^3x\epsilon^{\mu\rho\sigma}\bar{\Psi}_\mu D_\rho(\omega)D_\sigma(\omega)\epsilon = \frac{1}{8\kappa}\int d^3x\epsilon^{\mu\rho\sigma}\bar{\Psi}_\mu R_{\rho\sigma}^{mn}(\omega)\gamma_m\gamma_n\epsilon \tag{23}$$

We have used ,

$$[D_\rho(\omega)D_\sigma(\omega)]\epsilon = \frac{1}{4}R_{\rho\sigma}^{mn}(\omega)\gamma_m\gamma_n\epsilon \tag{24}$$

Now using (9-13) we get,

$$\epsilon^{\mu\nu\rho}R_{\nu\rho}^{mn}\gamma_{mn} = -\epsilon^{\mu\nu\rho}\epsilon_{mnr}R_{\nu\rho}^{mn}\gamma^r \tag{25}$$

$$= +6eR_{\nu\rho}^{mn}e_m^{[\mu}e_n^\nu e_r^{\rho]}\gamma^r \tag{26}$$

$$= 4ee_m^\mu(R_\rho^m - \frac{1}{2}Re_\rho^m)\gamma^r e_r^\rho \tag{27}$$

From the bosonic bit of action we get, using $\delta e = e^\nu_m \delta e^m_\nu$, we have,

$$\delta I_2 = -\frac{1}{8\kappa^2} R^{mn}_{\mu\nu}(\omega)\delta[ee^\nu_m e^\mu_,] = \frac{1}{4\kappa^2}\int d^3xe(R^m_\nu - \frac{1}{2}e^m_\nu R)\delta e^\nu_m \tag{28}$$

Thus we see that $\delta I_2 + \delta I_{3/2} = 0$. Hence we have shown how the action is invariant under supersymmetry.
Now we will calculate the local susy algebra, by requiring the closure of the supersymmetry commutator on the vielbein, $[\delta_1, \delta_2]e^m_\mu$.

Since we are using the 1.5 formulation, we split $\omega^{mn}_\mu(e,\psi)$ into the torsionless part $\omega^{mn}_\mu(e)$ and a torsion piece $\omega^{mn}_\mu(\psi)$

$$\omega^{mn}_\mu(e,\psi) = \omega^{mn}_\mu(e) + \omega^\psi_\mu \tag{29}$$

By varying the action with respect to ω we get the solution of the torsion piece as,

$$\omega^{mn}_\mu(\psi) = k^2(\bar{\psi}_\mu\gamma_m\psi_n - \bar{\psi}_\mu\gamma_n\psi_m + \bar{\psi}_\mu\gamma_m\psi_n) \tag{30}$$

Upon calculating we find the supersymmetry commutator on the vielbein as,

$$[\delta_1, \delta_2]e^m_\mu = (\partial_\mu\xi^\nu)e^m_\nu + \xi^\nu(\partial_\mu e^m_\nu) + [\xi^\nu\omega^{mn}_\nu(e,\psi)]e_{\mu m} + [\omega^m_{\mu s} - \omega^m_{s\mu}]\xi_s \tag{31}$$

We can easily identify the General Coordinate transformation, Local Lorentz and Supersymmetry pieces.

$$[\delta_1, \delta_2]e^m_\mu = \delta_{gc}(\xi^\nu) + \delta_{LL}(\xi^\nu\omega^{mn}_\nu(e,\psi)) + \delta_Q(-k\xi^\nu\psi_\nu) \tag{32}$$

6 Off-Shell SUSY

We saw that the off shell multiplet has an additional scalar S(auxiliary scalar). The addition to the action is,

$$I_s = \int d^3x(\frac{1}{2}eS^2) \tag{33}$$

Thus, the full off-shell action is given by,

$$S = \frac{-1}{8k^2}\int d^3xeR - \frac{1}{2}\int d^3xe\bar{\Psi}\gamma^{\mu\nu\rho}D_\nu(\omega)\Psi_\rho - \frac{1}{2}\int d^3x(eS^2) \tag{34}$$

The variation of Auxiliary S is given as,

$$\delta S = -\kappa\bar{\kappa}\gamma\cdot\psi S - \frac{2}{e}\epsilon^{\mu\rho\sigma}\bar{\epsilon}\gamma_\mu D_\rho(\omega)\psi_\sigma \tag{35}$$

The variation of ψ also gets an additional term $\delta_{additional}\psi = 2S\gamma_\mu\epsilon$. The off-shell commutation on veilbein is given by,

$$[\delta_1, \delta_2]e^m_\mu = \text{as before} + 4\kappa S\bar{\epsilon}_2\gamma^m\gamma_\mu\epsilon_1 - (1\leftrightarrow 2) \tag{36}$$

7 Matter Coupling and Massive Gravity stuff

There are three possible way of having massive gravity,

1. General Massive Gravity : This propagates two massive gravitons of helicity ± 2, generically with different masses. Topologically Massive gravity is a special case.

2. Scalar Massive Gravity : This is equivalent to a scalar field coupling the gravity

3. New Topologically Massive Gravity : This is a model in which the Einstein-Hilbert term is omitted. It involves a new scalar but turns out to propagate a single helicity 2 mode.

Following from [3] Scalar Massive gravity can be written in form of the supersymmetric non-linear sigma model is described by lagrangian,

$$L_{matter} = -\frac{1}{2} g_{ij}(\phi)\{\partial_\mu \phi^i \partial^\mu \phi^j + \bar{\chi}^i D(\Gamma)\chi^j\} + L_{\chi^4} \tag{37}$$

The covariant derivative is defined as,

$$D_\mu(\Gamma)\chi^i = \partial_\mu \chi^i + \Gamma^i_{jk}\partial_\mu \phi^j \chi^k \tag{38}$$

For N=1 connection would vanish and covariant derivative would just be a normal derivative since For N=1 the metric on the manifold is just a constant. L_{χ^4} is the quartic fermion bit.The full lagrangian is given by,

$$L = \frac{1}{\kappa}\{L_{sg} + L_{kin} + L_N + L_{\chi^4}\} \tag{39}$$

Here L_N includes terms required for supercovariance of χ_i field equation.

8 Conclusion

So in this note we presented the pure D=3 N=1 Supergravity and explicitly checked the supersymmetric invariance of the action. We also very briefly presented non-linear sigma model in D=3.

9 References

1. Horatiu Nastase,"Introduction to supergravity",arXiv:hep-th/1112.3502
2. F. Ruiz and P. Nieuwenhuizen, "Lectures on Supersymmetry and Supergravity in 2+1 Dimensions and Regularization of Supersymmetric Gauge Theories"
3. B. de Wit, A.K. Tollsten,"Locally Supersymmetric D=3 Non-Linear Sigma Model",arXiv:hep-th/9208074
4. Roel Andringa, Eric Bergshoeff, Mees de Roo,Olaf Hohm,Ergin Sezgin and Paul K Townsend,"Massive 3d Supergravity", arXiv:hep-th/0907.4658
5. S. Carlip, "Lectures on 2+1 Dimensional Gravity"