Coleman-Mandula no-go theorem

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1 Introduction

The Coleman-Mandula no-go theorem [1] is a powerful theorem that essentially states that, given some reasonable assumptions, the only possible Lie algebra of symmetry generators consist of the generators P_μ and $J_{\mu\nu}$ of the Poincar group, and internal symmetry generators who commute with the Poincar group. The theorem tells us, that all generators of internal symmetries (gauge symmetries in the standard model) will commute with the generators of the Poincar group (space-time symmetries). Another way of stating is "It is a theorem on the impossibility of combining space-time and internal symmetries in any but a trivial way."

What I have shown here is a filled out version of the proof of the theorem heavily taken from Weinberg's Supersymmetry [2].

1.1 The theorem

We will now directly cite the theorem from [1].

Theorem: Let $\mathcal{G}$ be a connected symmetry group of the S-matrix, and let the following five conditions hold:

1. **(Lorentz invariance.)** $\mathcal{G}$ contains a subgroup locally isomorphic to $\mathcal{P}$(Poincar algebra).
2. **(Particle-finiteness.)** All particle types correspond to positive-energy representations of $\mathcal{P}$. For any finite M, there are only a finite number of particle types with mass less than M.
3. **(Weak elastic analyticity.)** Elastic-scattering amplitudes are analytic functions of centerof-mass energy, s, and invariant momentum transfer, t, in some neighborhood of the physical region, except at normal thresholds.
4. **(Occurrence of scattering.)** Let $|p\rangle$ and $|p'\rangle$ be any two one-particle momentum eigenstates, and let $|pp'\rangle$ be the two-particle state made from these. Then,

   $$T|pp'\rangle \neq 0$$

   except perhaps for certain isolated values of s. Phrased briefly, at almost all energies, any two plane waves scatter.
5. **(An ugly technical assumption.)** The generators of $\mathcal{G}$, considered as integral operators in momentum space, have distributions for their kernels.

Then, $\mathcal{G}$ is locally isomorphic to the direct product of an internal symmetry group and the Poincar group.

In this proof we will deal with symmetry generators. So given the above assumptions the statement of the theorem reads that given symmetry generators $\mathcal{B}$ not in the Poincar group we have,

$$[\mathcal{B}, \mathcal{P}] = 0 \tag{1}$$

$\mathcal{B}$ commutes with all generators of $\mathcal{P}$. We will use the same symbol for group and the algebra.

1.2 Gameplan

Let $\mathcal{G}$ be a symmetry of S-matrix.

1. We begin with a subalgebra $\mathcal{B}$ of $\mathcal{G}$ called B_α which commutes with P_μ and show that this is finite dimensional.
2. Being finite dimensional it can be decomposed as a direct sum of semi-simple lie algebra and some number U(1)s. We show that the U(1) generators commute with Lorentz generators.
3. We take the remaining semi-simple compact Lie algebra and show that it also commutes with Lorentz generator thus proving that $\mathcal{B}$ is an internal symmetry.
4. Finally we deal with $\mathcal{A}$, the remaining part of $\mathcal{G}$ that does not commute with P_μ and show that it is just the sum of some internal symmetry generator and Lorentz generator

1.3 S-matrix crash course

In scattering we consider physical states to be asymptotic ie in the distant past $(t \to -\infty)$ are denoted $|\text{in}\rangle$, and states in distant future $(t \to +\infty)$ are denoted as $|\text{out}\rangle$. These states form a complete set of states and they are orthonormal,

$$\langle m, \text{in}|n, \text{in}\rangle = \langle m, \text{out}|n, \text{out}\rangle = \delta_{mn} \tag{2}$$

The matrix elements of S give the overlap between configurations of in and out states,

$$S(\psi \to \phi) = \langle \phi, \text{out}|\psi, \text{in}\rangle = \langle \phi, \text{out}|S|\psi, \text{out}\rangle = \langle \phi, \text{in}|S|\psi, \text{in}\rangle \tag{3}$$

S satisfies the following relations,

$$S = \sum_m |m, \text{in}\rangle\langle m, \text{out}| \tag{4}$$

$$S^\dagger = \sum_m |m, \text{out}\rangle\langle m, \text{in}| \tag{5}$$

$$S^\dagger S = SS^\dagger = 1 \tag{6}$$

S can be decomposed as,

$$S = 1 + iT \tag{7}$$

here 1 refers to particles not interacting, while T is the connected part of S matrix. In terms of invariant matrix element $\mathcal{M}$ we have,

$$\langle p'm', q'n'|iT|pm, qn\rangle = (2\pi)^4\delta^4(p' + q' - p - q)i(\mathcal{M}(p', q'; p, q))_{m'n',mn} \tag{8}$$

So we have,

$$S(pm, qn \to p'm, q'n')_{\text{connected}} = (2\pi)^4\delta^4(p' + q' - p - q)i(\mathcal{M}(p', q'; p, q))_{m'n',mn} \tag{9}$$

1.4 Useful Identities

Here we will list down some useful identities used in this note. The relativistic normalization of two particle states is given by,

$$\langle p'q'|pq\rangle = 2E_p 2E_q (2\pi)^3(\delta^3(\mathbf{p} - \mathbf{p}')\delta^3(\mathbf{q} - \mathbf{q}')\delta^3(\mathbf{p} - \mathbf{q}')\delta^3(\mathbf{q} - \mathbf{p}')) \tag{10}$$

Non-relativistic one-particle identity operator,

$$(1)_{\text{1-particle}} = \int \frac{d^3p}{(2\pi)^3}|p\rangle\frac{1}{2E_p}\langle p| \tag{11}$$

2 $\mathcal{B}$

We begin by considering the sub-algebra $\mathcal{B}$ of $\mathcal{G}$ which consists of symmetry generator B_α that commute with four-momentum P_μ.

$$[B_\alpha, P_\mu] = 0 \tag{12}$$

B_α has a momentum-dependent representation b_α when acting upon single particles states,

$$B_\alpha|pm\rangle = (b_\alpha(p))_{m'm}|pm'\rangle \tag{13}$$

It acts on multi-particle state as a tensor product of one-particle states,

$$B_\alpha|p, m; q, n; \cdots\rangle = (b_\alpha(p))_{m'm}|p, m'; q, n; \cdots\rangle + (b_\alpha(q))_{n'n}|p, m; q, n'; \cdots\rangle + \cdots \tag{14}$$

The generators B_α obey a lie algebra,

$$[B_\alpha, B_\beta] = iC^\gamma_{\alpha\beta}B_\gamma \tag{15}$$

$$[B_\alpha, B_\beta]|p, m\rangle = iC^\gamma_{\alpha\beta}(b_\gamma(p))_{m'm}|p, m'\rangle \tag{16}$$

The matrix elements obey the same Lie algebra,

$$[b_\alpha(p), b_\beta(p)] = iC^\gamma_{\alpha\beta}b_\gamma(p) \tag{17}$$

The action of B_α on a two particle state is given as,

$$B_\alpha|p, m; q, n\rangle = (b_\alpha(p, q))_{m'n',mn}|p, m'; q, n'\rangle \tag{18}$$

the matrix representation of $b_\alpha(p, q)$ acting on two particle state is defined as,

$$(b_\alpha(p, q))_{m'n',mn} = (b_\alpha(p))_{m'm}\delta_{n'n} + (b_\alpha(q))_{n'n}\delta_{m'm} \tag{19}$$

So we have seen that B_α and $b_\alpha(p)$ have a homeomorphism between them (since they satisfy the same lie algebra). What we need is an isomorphism so that there is a one to one correspondence. Why do we need this ? The answer is , **any Lie algebra of finite Hermitian matrices (like $b_\alpha(p)$) must be a direct sum of semi-simple Lie algebra and U(1) algebras** [3]. If we show the isomorphism between B_α and $b_\alpha(p)$ then the B_α algebra would also have to be a direct sum of semi-simple Lie algebra and U(1) algebras. So how do we show this isomorphism ?

If for some four momentum p, we find some coefficients c^α such that,

$$c^\alpha b_\alpha(p) = 0 \tag{20}$$

it means that the $b_\alpha(p)$ are not linearly dependent. Showing $c^\alpha b_\alpha(k) = 0$ for all four-momentum k is equivalent to the condition $c^\alpha B_\alpha = 0$, then b_α and B_α share the same degeneracy and hence are isomorphic.

Now if the B_α are symmetry generators, they must commute with the S-matrix.

$$[B_\alpha, S] = 0 \tag{21}$$

We consider an elastic 2-2 scattering $(p, q \to p'q')$ with $p + q = p' + q'$, with p, q, p', q' all on mass-shell, $p^2 = p'^2$ and $q^2 = q'^2$. So for this scattering situation we have,

$$\langle p', m'; q', n'|[B_\alpha, S]|p, m; q, n\rangle = 0 \tag{22}$$

We will make use of the particle exchange property,

$$B_\alpha|pm, qn\rangle = (-1)^{\text{spin}}B_\alpha|qn, pm\rangle = (-1)^{\text{spin}}(b_\alpha(q, p))_{n'm',nm}|qn', pm'\rangle \tag{23}$$

This gives us

$$(-1)^{\text{spin}}(b_\alpha(q,p))_{m'n',nm}|qm',pn'\rangle = ((-1)^{\text{spin}})^2(b_\alpha(q,p))_{m'n',nm}|pn',qm'\rangle \tag{24}$$

This gives us,

$$B_\alpha|pm,qn\rangle = (b_\alpha(p,q))_{n'm',mn}|pn',qm'\rangle = (b_\alpha(q,p))_{m'n',nm}|pn',qm'\rangle \tag{25}$$

Using (22),(8),(9) and (10) we have,

$$b_\alpha(p',q')\mathcal{M}(p',q';p,q) = \mathcal{M}(p',q';p,q)b_\alpha(p,q) \tag{26}$$

This says that for two particle scattering that conserves the total momentum $p+q \to p'+q'$, the matrix representations of B_α are related by similarity transformation.

$$b_\alpha(p'q') = S(p',q';p,q)b_\alpha(p,q)S^{-1}(p',q';p,q) \tag{27}$$

If we can find coefficients c^α and four-momenta p,q such that,

$$c^\alpha b_\alpha(p,q) = 0 \tag{28}$$

From this we can conclude that for any four momenta on the mass shell p,q,p' and q' which satisfy $p+q \to p'+q'$, using similarity transform we can obtain,

$$\begin{aligned} c^\alpha b_\alpha(p'q') &= S(p',q';p,q)c^\alpha b_\alpha(p,q)S^{-1}(p',q';p,q) \\ &= 0 \end{aligned} \tag{29}$$

However this does not tell us, $c^\alpha b_\alpha(p') = c^\alpha b_\alpha(q') = 0$. This is the condition we need for isomorphism. We get,

$$c^\alpha(b_\alpha(p',q')) = c^\alpha[(b_\alpha(p'))_{m'm}\delta_{n'n} + (b_\alpha(q'))_{n'n}\delta_{m'm}] = 0 \tag{30}$$

so we get,

$$c^\alpha(b_\alpha(p'))_{m'm}\delta_{n'n} = -(b_\alpha(q'))_{n'n}\delta_{m'm} \tag{31}$$

This tells us that $c^\alpha b_\alpha(p')$ and $c^\alpha b_\alpha(q')$ are proportional to identity matrix with opposite sign. Considering the trace of two particle matrices,

$$\begin{aligned} Tr[b_\alpha(p',q')] =& Tr[S(p',q';p,q)b_\alpha(p,q)S^{-1}(p',q';p,q)] \\ &= Tr[b_\alpha(p,q)] \end{aligned} \tag{32}$$

Trace does not change under similarity transformation.

$$\begin{aligned} Tr[(b_\alpha(p,q))_{m'n',mn}] &= Tr[(b_\alpha(p))_{m'm}\delta_{n'n} + (b_\alpha(q))_{n'n}\delta_{m'm}] \\ &= N(m_q)tr[(b_\alpha(p))_{m'm}] + N(m_p)tr[(b_\alpha(p))_{n'n}] \end{aligned} \tag{33}$$

N(m) is the multiplicity of particles with mass m. Now using (32) we can write,

$$N(m_q)tr[(b_\alpha(p'))] + N(m_p)tr[(b_\alpha(q'))] = N(m_q)tr[(b_\alpha(p))] + N(m_p)tr[(b_\alpha(p))] \tag{34}$$

this should hold for all mass-shell momenta satisfying $p+q \to p'+q'$. We get the following solution,

$$\frac{tr[b_\alpha(p)]}{N(m_p)} = a_\alpha^\mu P_\mu \tag{35}$$

here a_α^μ are constants.

Now we define new symmetry generators from B_α by subtracting terms linear in P_μ. **The motivation here is to make the matrix representation traceless**.

$$\boxed{B_\alpha^\# := B_\alpha - a_\alpha^\mu P_\mu} \tag{36}$$

The action of this on one particle states is given as,

$$B_\alpha^{\#}|pm\rangle = ((b_\alpha(p))_{m'm} - a_\alpha^\mu p_\mu \delta_{m'm})|pm'\rangle \tag{37}$$

$$\equiv b_\alpha^{\#}(p)_{m'm}|pm'\rangle \tag{38}$$

It can be easily checked from its definition that,

$$tr\left[(b_\alpha^{\#}(p))_{m'm}\right] = 0 \tag{39}$$

Since P_μ commutes with B_α, then it also commutes with $B_\alpha^{\#}$,

$$[P_\mu, B_\alpha^{\#}] = 0 \tag{40}$$

Trace of commutator of traceless matrices is 0.

$$tr[b_\alpha^{\#}(p), b_\beta^{\#}(p)] = itr[C_{\alpha\beta}^\gamma(b_\gamma^{\#}(p) + a_\gamma^\mu p_\mu)] = 0 \tag{41}$$

This implies,

$$C_{\alpha\beta}^\gamma a_\gamma^\mu = 0 \tag{42}$$

Using this we can show that $B_\alpha^{\#}$ satisfy a Lie algebra,

$$[B_\alpha^{\#}, B_\beta^{\#}] = iC_{\alpha\beta}^\gamma B_\gamma^{\#} \tag{43}$$

It is also a generator of symmetry ie,

$$\langle p', m'; q', n'|[B_\alpha^{\#}, S]|p, m; q, n\rangle = 0 \tag{44}$$

Repeating the previous analysis we again find,

$$b_\alpha^{\#}(p'q') = S(p', q'; p, q)b_\alpha^{\#}(p, q)S^{-1}(p', q'; p, q) \tag{45}$$

$b_\alpha^{\#}(p, q)$ are matrices representing the action of $B_\alpha^{\#}$ on two particle states.

$$B_\alpha^{\#}|p, m; q, n\rangle = (b_\alpha^{\#}(p, q))_{m'n',mn}|p, m^{'}; q, n^{'}\rangle \tag{46}$$

where

$$(b_\alpha^{\#}(p, q))_{m'n',mn} = (b_\alpha^{\#}(p))_{m'm}\delta_{n'n} + (b_\alpha^{\#}(q))_{n'n}\delta_{m'm} \tag{47}$$

If we have,

$$c^\alpha b_\alpha^{\#}(p, q) = 0 \implies c^\alpha b_\alpha^{\#}(p', q') = 0 \tag{48}$$

This means that, since $\mathcal{M}(p', q'; p, q)$ is a non-singular and analytic matrix, if we find coefficients c^α that satisfy the first criteria for some fixed momenta p and q, then the second condition is implied for all p' and q' on mass shell and also satisfy $p' + q' = p + q$ such that the similarity transform (45) exists. From the second condition we get,

$$c^\alpha(b_\alpha^{\#}(p'))_{m'm}\delta_{n'n} = -c^\alpha(b_\alpha^{\#}(q'))_{n'n}\delta_{m'm} \tag{49}$$

Since $b_\alpha^{\#}$ are traceless , we have ,

$$c^\alpha b_\alpha^{\#}(p') = c^\alpha b_\alpha^{\#}(q') = 0 \tag{50}$$

So what we have found is that, if we can find a set of ceofficients such that $c^\alpha b_\alpha^{\#}(p, q) = 0$ for some fixed mass shell four -momentum p and q, then $c^\alpha b_\alpha^{\#}(p') = c^\alpha b_\alpha^{\#}(q') = 0$ for all p' and q' on the same mass shell that satisfy four-momentum conservation $p' + q' = p + q$.

$$c^\alpha b_\alpha^{\#}(p, q) = c^\alpha b_\alpha^{\#}(p', q') = 0 \tag{51}$$

from this we know that,

$$c^\alpha b^{\#}_\alpha(p) = c^\alpha b^{\#}_\alpha(q) = c^\alpha b^{\#}_\alpha(p') = c^\alpha b^{\#}_\alpha(q') = 0 \tag{52}$$

From this and (51) we can deduce,

$$c^\alpha b_\alpha(p, q') = 0 \tag{53}$$

Using a similarity transform acting on states of total momentum $p + q'$ we have,

$$c^\alpha b_\alpha(k, p + q' - k) = 0 \tag{54}$$

This similarity transform only exists when both k and $p + q' - k$ are both on mass shell. We first look at the scattering $p + q \to p'(= p + q - q'), q'$. For the collision to be elastic we have,

$$m_p^2 = (p + q - q')^2 \tag{55}$$

$$m_q^2 = q'^2 \tag{56}$$

These equations remove two degrees of freedom from q'. Next we consider scattering $p, q' \to k, (p + q' - k)$. From this we have,

$$m_p^2 = k^2 \tag{57}$$

which removes one degree of freedom from k. We also have,

$$m_q^2 = (p + q' - k)^2 \tag{58}$$

We still have enough freedom in q such that k is unconstrained. So we are free to choose $\mathbf{k}$, the three-vector component of k in any way we like. So we have shown that if for some fixed mass-shell momenta p and q

$$c^\alpha b^{\#}_\alpha(p, q) = 0 \tag{59}$$

then we have for almost all k,

$$c^\alpha b^{\#}_\alpha(k) = 0 \tag{60}$$

This is unconstrained by kinematics because we are free to choose the 3-vector $\mathbf{k}$.

Now suppose for some mass-shell momenta p and q that $c^\alpha b^{\#}_\alpha(p, q) = 0$. What happens if for some k_0 we have $c^\alpha b^{\#}_\alpha(k_0) \neq 0$? If this was the case, a scattering process where particles with four-momenta k_0 and k and scatter into particles of four momenta k' and k'' will be forbidden by the symmetry generated by $B^{\#}_\alpha$ since if the symmetry allowed such a scattering process , a similarity transform would exist between $b^{\#}_\alpha(k_0, k)$ and $b^{\#}_\alpha(p, q)$ where $c^\alpha b^{\#}_\alpha(p, q) = 0$. Our initial assumption was that scattering amplitudes are analytic function of scattering angle at almost all energies. What this means is that the scattering amplitude to a particle with momentum k_0 cannot jump to zero under the symmetry imposed by $B^{\#}_\alpha$ in any analytic way, so the existence of such a state is in contradiction with one of our assumptions. So from this we conclude that if we have (59) then we have ,

$$c^\alpha b^{\#}_\alpha(k) = 0 \tag{61}$$

for all k and hence,

$$c^\alpha B^{\#}_\alpha = 0 \tag{62}$$

Thus the mapping that takes $B^{\#}_\alpha$ into $b^{\#}_\alpha(p, q)$ is therefore an isomorphism. From the definition of $B^{\#}_\alpha$ **it is clear that there is an isomorphism between** B_α **and** $b^{\#}_\alpha(p, q)$. Looking at the action of $B^{\#}_\alpha$ on two particle states,

$$(b^{\#}_\alpha(p, q))_{m'n',mn} = (b^{\#}_\alpha(p))_{m'm}\delta_{n'n} + (b^{\#}_\alpha(q))_{n'n}\delta_{m'm} \tag{63}$$

we see that for any given m,n the number of independent $b^{\#}_\alpha(p, q)$ cannot exceed $N(m_p)N(m_q)$, hence it is finite dimensional. Due to the isomorphism it means that there are at most a finite number of independent symmetry generators B_α. Thus we have shown that B_α must be finite dimensional. Using theorem proved in chapter 15 of [3] , any Lie algebra of finite hermitian matrices like b_α can be written as direct sum of compact semi-simple Lie algebra and U(1) algebras. Because of the isomorphism we will split B_α into the U(1)s (B_i) and semi-simple Lie algebra (B_α).

3 Dealing with U(1)s

We know that $B_\alpha^\#$ commute with P_μ.

$$[P_\mu, B_\alpha^\#] = 0 \tag{64}$$

From Lorentz algebra we also know that,

$$[J, P_\mu] \sim (\text{linear combination of } P_\mu) \tag{65}$$

Using the Jacobi identity we find,

$$[P_\mu, [J, B_\alpha^\#]] + [J, [B_\alpha^\#, P_\mu]] + [B_\alpha^\#, [P_\mu, J]] = 0 \tag{66}$$

This gives us,

$$[P_\mu, [J, B_\alpha^\#]] = 0 \tag{67}$$

Since we have defined all generators that commute with P_μ consists of generators B_α, it follows that $[J, B_\alpha^\#]$ must then be a linear combination of $B_\alpha^\#$.

$$[J, B_\alpha^\#] = c_\alpha^\beta B_\beta^\# \tag{68}$$

We denote the generator of U(1) Lie algebra as $B_i^\#$ in the algebra of $B_\alpha^\#$. These U(1) generators must commute with all of $B_\alpha^\#$ since it is a U(1) subalgebra of $\mathcal{B}$. Thus we have,

$$[B_i^\#, [J, B_i^\#]] \sim [B_i^\#, B_\alpha^\# + B_i^\#] = [B_i^\#, B_i^\#] = 0 \tag{69}$$

We use,

$$J|p, m; q, n\rangle = \sigma(m, n)|p, m; q, n\rangle \tag{70}$$

and calculate the expectation value. We get,

$$\begin{aligned} 0 =& \langle p, m; q, n|[B_i^\#, [J, B_i^\#]]|p, m; q, n\rangle \\ & \langle p, m; q, n|(2B_i^\# J B_i^\# - J B_i^\# B_i^\# - B_i^\# B_i^\# J)|p, m; q, n\rangle \\ & 2\langle p, m; q, n|(B_i^\# J B_i^\# - J B_i^\# B_i^\#)|p, m; q, n\rangle \\ & 2(\sigma(m', n') - \sigma(m, n))|(b_i^\#(p, q))_{m'n', mn}|^2 \end{aligned} \tag{71}$$

here we have used the hermitian nature of J, $J^\dagger = J$. So we see that for any m, n, m' and n' for which $\sigma(m', n') \neq \sigma(m, n)$, we have $(b_i^\#(p, q))_{m'n', mn}$ has to vanish. From the isomorphism of $b_i^\#(p, q)$ and $B_i^\#$ we have,

$$[B_i^\#, J] = 0 \tag{72}$$

There is a slight subtlety here, we have chosen J to be spatial rotations in the x-y plane (ie around the z axis) so our J is basically J_z . By choosing appropriate directions of the 3-vectors $\mathbf{p}$ and $\mathbf{q}$ we can conclude that $[B_i^\#, J] = 0$ for all J(spatial rotations).

A similar analysis can be carried out for boost generators (which we will not do it here) which again tells us that boost commutes with $B_i^\#$. Thus we have,

$$\boxed{[B_i^\#, J_{\mu\nu}] = 0} \tag{73}$$

$B_i^\#$ commutes with all generators of Lorentz group. This tells us that the matrix representation $b_i^\#(p)_{n'n}$ is independent of particle momentum and act as identity matrices on spin indices. **We conclude that $B_i^\#$ are the generators of ordinary internal symmetry.**

4 Remaining semi-simple compact Lie algebra

Now we have to deal with the remaining $B_\alpha^{\#}$, the generators of a compact lie algebra. For this we go back to B_α. If these generators commute with P_μ, we have

$$[P_\mu, B_\alpha] = iC^\alpha_{\mu\beta}B_\alpha = 0 \tag{74}$$

so we have,

$$C^\alpha_{\mu\beta} = 0 \tag{75}$$

Under lorentz transformation we have,

$$B_\alpha \to U(\Lambda)B_\alpha U^{-1}(\Lambda) \tag{76}$$

This should also commute with P_μ

$$[U(\Lambda)B_\alpha U^{-1}(\Lambda), P_\mu] = 0 \tag{77}$$

Since we stated in the beginning that all symmetry generators that commute with P_μ are spanned by generators B_α we have,

$$U(\Lambda)B_\alpha U^{-1}(\Lambda) = D^\beta{}_\alpha(\Lambda)B_\beta \tag{78}$$

where $D(\Lambda)$ is a representation of homogeneous Lorentz group. The Lorentz group has both finite-dimensional and infinite-dimensional representations. However, it is non-compact, therefore its finite-dimensional representations are not unitary (the generators are not Hermitian). The only finite dimensional representation is the trivial one.

$$D(\Lambda) = 1 \tag{79}$$

We get,

$$U(\Lambda)B_\alpha U^{-1}(\Lambda) = D^\beta{}_\alpha(\Lambda)B_\beta = B_\alpha \tag{80}$$

Thus, we have

$$\boxed{[B_\alpha, U(\Lambda)] = 0} \tag{81}$$

So B_α commutes with Lorentz group. **So we have shown that all generator B_α of $\mathcal{B}$ commute with P_μ (from definition of $\mathcal{B}$) and J(as shown) and therefore with all generators of Poincar group. Thus they are internal symmetries.**

5 $\mathcal{A}$ - sub-algebra that does not commute with P_μ

Let us now look at possible symmetry generators that do not commute with P_μ. This is the subset $\mathcal{A}$ of $\mathcal{G}$. The action of a general symmetry generator, which we will call A_α, on a one-particle state $|p, n\rangle$ of four momentum p is given as,

$$A_\alpha|p, n\rangle = \int d^4p'(\mathcal{A}_\alpha(p', p)_{n'n})|p', n'\rangle \tag{82}$$

Being a symmetry generator $\mathcal{A}_\alpha$ (also called the 'kernel'), would vanish unless both p' and p are on mass-shell. Now since A_α is a symmetry generator then so is,

$$A^f_\alpha|p, n\rangle = \int d^4x \exp(iP\cdot x)A_\alpha \exp(-iP\cdot x)f(x) \tag{83}$$

$f(x)$ is a function we can choose as we like. By acting this on a one particle state we find,

$$\int d^4p'\tilde{f}(p'-p)(\mathcal{A}_\alpha(p', p))_{n'n}|p', n'\rangle \tag{84}$$

here $\tilde{f}(k)$ is the Fourier transform,

$$\tilde{f}(k) = \int d^4x \exp(ik \cdot x) f(x) \tag{85}$$

Now, let us suppose that there is a pair of mass shell four-momenta p_1 and $p_1 + \Delta$ with $\Delta \neq 0$. For a two particle scattering process with four-momenta satisfying $p_1 + q_1 \to p_2 + q_2$, then in general, $q_1 + \Delta$,$p_2 + \Delta$ and $q_2 + \Delta$ will not be mass shell. If we take $\tilde{f}(k)$ to vanish outside a sufficiently small region around Δ, then A^f_α will annihilate all one-particle states with four-momentum q_1, p_2 and q_2 but not the one-particle states with four-momentum p_1.

$$A^f_\alpha |p_1, n\rangle = \int d^4p'_1 \tilde{f}(p'_1 - p_1)(\mathcal{A}_\alpha(p'_1, p_1))_{n'n}|p'_1, n'\rangle = \tilde{f}(\Delta)(\mathcal{A}_\alpha(p_1 + \Delta, p_1))_{n'n}|p_1 + \Delta, n'\rangle \tag{86}$$

$$A^f_\alpha |q_1, n\rangle = \int d^4q'_1 \tilde{f}(q'_1 - q_1)(\mathcal{A}_\alpha(q'_1, q_1))_{n'n}|q'_1, n'\rangle = \tilde{f}(\Delta)(\mathcal{A}_\alpha(q_1 + \Delta, q_1))_{n'n}|q_1 + \Delta, n'\rangle \tag{87}$$

Since Kernel vanishes whenever the momentum are not on mass shell we get,

$$A^f_\alpha |q_1, n\rangle = A^f_\alpha |p_2, n\rangle = A^f_\alpha |q_2, n\rangle = 0 \tag{88}$$

$$A^f_\alpha |p_1, n\rangle \neq 0 \tag{89}$$

Now this is BAD, because we assumed that scattering occurs at almost all energies. With what we have just shown, the symmetry generated by A^f_α forbids a process to have the kinematics $p_1 + q_1 \to p_2 + q_2$. This would mean that scattering does not occur at almost all energies , contradicting our initial assumption. This porblem can be averted if A_α commutes with four-momentum P_μ,

$$A^f_\alpha |p, n\rangle = \int d^4x \exp(iP \cdot x) A_\alpha \exp(-iP \cdot x) f(x)|p, n\rangle \tag{90}$$

$$= (\int d^4x f(x)) A_\alpha |p, n\rangle \propto A_\alpha |p, n\rangle \tag{91}$$

So $\tilde{f}(x)$ does not appear in action and cannot be appropriately chosen to cause problems. However we cannot use this option because if A_α commuted with P_μ then it would just be a linear combination of B_α.

Instead of A_α commuting with P_μ we consider the kernel to be proportional to momentum space delta function.

$$(\mathcal{A}_\alpha(p', p))_{n'n} = \delta^4(p' - p)(a^0_\alpha(p', p))_{n'n} \tag{92}$$

With this definition the action of A^f_α on one particle state,(82) becomes,

$$A^f_\alpha |p, n\rangle = \int d^4p' \tilde{f}(p' - p)\delta^4(p' - p)(a^0_\alpha(p', p))_{n'n}|p', n'\rangle = \tilde{f}(0)(a^0_\alpha(p))_{n'n}|p, n'\rangle \tag{93}$$

As we can see the arbitrary function $\tilde{f}(0)$ is independent of the state it acts on. Thus if A^f_α does not annihilate a state with momentum p, it would also not annihilate state with momentum p'. So the symmetry generated by A_α would allow a scattering process with kinematics, $p_1 + q_1 \to p_2 + q_2$ for all the momentums on mass shell. Now we utilize the last assumption of the Coleman-Mandula theorem and consider the kernel $A_\alpha(p', p)$ to be a distribution which means that it contains objects proportional to delta functions and finite derivatives of delta functions. Thus with this in mind we can give an expansion to the kernel with D derivatives,

$$(\mathcal{A}_\alpha(p', p))_{n'n} = (a^0_\alpha(p', p))_{n'n}\delta^4(p' - p) + (a^1_\alpha(p', p))^{\mu_1}_{n'n} \frac{\partial}{\partial p'^{\mu_1}} \delta^4(p' - p) \tag{94}$$

$$+ \cdots + (a^D_\alpha(p', p))^{\mu_1 \cdots \mu_D}_{n'n} \frac{\partial^D}{\partial p'^{\mu_1} \cdots \partial p'^{\mu_D}} \delta^4(p' - p) \tag{95}$$

Now if i consider the first derivative term,

$$(a^1_\alpha(p',p))^{\mu_1}_{n'n}\frac{\partial}{\partial p'^{\mu_1}}\delta^4(p'-p) \tag{96}$$

Its action on a one particle state gives(using (96) and (82)),

$$(A^f_\alpha|p,n\rangle)|_{\text{first derivative}} = -\tilde{f}(0)\frac{\partial}{\partial p^{\mu_1}}((a^1_\alpha(p))^{\mu_1}_{n'n}|p,n'\rangle) \tag{97}$$

Taking the first term and the first derivative and expanding a bit we have,

$$A^f_\alpha|p,n\rangle = \tilde{f}(0)\left((a^0_\alpha(p))_{n'n}) - \frac{\partial}{\partial p^{\mu_1}}(a^1_\alpha(p))^{\mu_1}_{n'n} - (a^1_\alpha(p',p))^{\mu_1}_{n'n}\frac{\partial}{\partial p^{\mu_1}}|p,n'\rangle\right) + \text{higher derivative} \tag{98}$$

Now we define,

$$(a'^0_\alpha(p))_{n'n} \equiv \tilde{f}(0)\left((a^0_\alpha(p))_{n'n} - \frac{\partial}{\partial p^{\mu_1}}(a^1_\alpha(p))^{\mu_1}_{n'n}\right) \tag{99}$$

We can generalize this iteratively for the higher derivatives and then finally we obtain,

$$A^f_\alpha|p,n\rangle = \left((a'^0_\alpha(p))_{n'n} + (a'^1_\alpha(p))^{\mu_1}_{n'n}\frac{\partial}{\partial p^{\mu_1}} + \cdots + (a'^D_\alpha(p))^{\mu_1\cdots\mu_D}_{n'n}\frac{\partial^D}{\partial p^{\mu_1}\cdots\partial p^{\mu_D}}\right)|p,n'\rangle \tag{100}$$

Similarly we have,

$$A_\alpha|p,n\rangle = \left((a'^0_\alpha(p))_{n'n} + (a'^1_\alpha(p))^{\mu_1}_{n'n}\frac{\partial}{\partial p^{\mu_1}} + \cdots + (a'^D_\alpha(p))^{\mu_1\cdots\mu_D}_{n'n}\frac{\partial^D}{\partial p^{\mu_1}\cdots\partial p^{\mu_D}}\right)|p,n'\rangle \tag{101}$$

The coefficents here are in general different from the ones in A^f_α because of $\tilde{f}(0)$, since A_α does not depend on the choice of $\tilde{f}(k)$.

Now since A_α are symmetry generators, it must contain B_α as a subset since B_α commute with P_μ. B_α act as matrices on states (18) instead of polynomials (101). Therefore B_α can be formed from D-fold commutator of momentum and A_α.

$$B^{\mu_1\mu_2\cdots\mu_D}_\alpha = [P^{mu_1},[P^{\mu_2},\cdots[P^{\mu_D},A_\alpha]]\cdots] \tag{102}$$

To show how this works , lets take A_α with D=1 acting on a one particle state,

$$\begin{aligned}B^\mu_\alpha|p,n\rangle =&[P^\mu,A_\alpha]|p,n\rangle\\ &= [P^\mu,(a^0_\alpha(p))_{n'n} + (a^1_\alpha(p))^\nu_{n'n}\frac{\partial}{\partial p^\nu}|p,n'\rangle\\ &= -(a'^1_\alpha(p))^\mu_{n'n}|p,n'\rangle\end{aligned} \tag{103}$$

So we have,

$$B^\mu_\alpha|p,n\rangle = (b_\alpha(p))^\mu_{n'n}|p,n'\rangle = -(a'^1_\alpha(p))^\mu_{n'n}|p,n'\rangle \tag{104}$$

Now we need to show that $B^{\mu_1\cdots\mu_D}_\alpha$ commutes with momentum. Basically we are trying to show (102) is the same as $\mathcal{B}$. We can show that by taking commutation with momentum between states of different momentum and then taking the limit $p_1\to p_2$.

$$\begin{aligned}\langle p_2|[B^{\mu_1}_\alpha,P^\mu]|p_1\rangle &= \langle p_2|[[P^{\mu_1},A_\alpha],P^\mu]|p_1\rangle\\ &= -(p_2-p_1)^{\mu_1}(p_2-p_1)^\mu\langle p_2|A_\alpha|p_1\rangle\\ &= -(p_2-p_1)^{\mu_1}(p_2-p_1)^\mu\int d^4p'\langle p_2|\mathcal{A}_\alpha(p',p_1)|p'\rangle\\ &= -(p_2-p_1)^{\mu_1}(p_2-p_1)^\mu(a'^0_\alpha(p_1)+a^1_\alpha(p_1)^\nu\frac{\partial}{\partial p'^\nu})\langle p_2|p_1\rangle\\ &= -(p_2-p_1)^{\mu_1}(p_2-p_1)^\mu(a'^0_\alpha(p_1)+a^1_\alpha(p_1)^\nu\frac{\partial}{\partial p'^\nu})\delta^4(p_2-p_1)\end{aligned} \tag{105}$$

Here $a'^0_\alpha(p_1)$ is as defined previously. Generalizing this we get,

$$\langle p_2|[[P^{\mu_1},[P^{\mu_2}\cdots[P^{\mu_D},A_\alpha]]\cdots],P^\mu]|p_1\rangle = -(p_2-p_1)^\mu(p_2-p_1)^{\mu_1}\cdots(p_2-p_1)^{\mu_D}\langle p_2|A_\alpha|p_1\rangle \tag{106}$$

$$\langle p_2|A_\alpha|p_1\rangle \propto (a'^0_\alpha(p_1)+a^1_\alpha(p_1)^\nu\frac{\partial}{\partial p'^\nu}+\cdots+a'^D_\alpha(p_1)^{\nu_1\cdots\nu_D}\frac{\partial^D}{\partial p'^{\nu_1}\cdots\partial p'^{\nu_D}})\delta^4(p_2-p_1) \tag{107}$$

Combining the two we get,

$$\begin{aligned}\langle p_2|[B^{\mu_1\mu_2\cdots\mu_D}_\alpha,P^\mu]|p_1\rangle &= -(p_2-p_1)^\mu(p_2-p_1)^{\mu_1}\cdots(p_2-p_1)^{\mu_D}\langle(a'^0_\alpha(p_1)\\ &+a^1_\alpha(p_1)^\nu\frac{\partial}{\partial p'^\nu}+\cdots+a'^D_\alpha(p_1)^{\nu_1\cdots\nu_D}\frac{\partial^D}{\partial p'^{\nu_1}\cdots\partial p'^{\nu_D}})\delta^4(p_2-p_1)\end{aligned} \tag{108}$$

Because the delta function will take $p_2 \to p_1$ this would cause the prefactors to vanish, thus the commutation vanishes. This indicates that $B^{\mu_1\mu_2\cdots\mu_D}_\alpha$ is in $\mathcal{B}$.

We know that action of B_α on one particle states can be written as,

$$(b_\alpha(p))_{n'n} = (b^\#_\alpha)_{n'n} + a^\mu_\alpha p_\mu \delta_{n'n} \tag{109}$$

This can be generalized for $B^{\mu_1\mu_2\cdots\mu_D}_\alpha$ as,

$$(b_\alpha(p))^{\mu_1\cdots\mu_D}_{n'n} = (b^\#_\alpha)^{\mu_1\cdots\mu_D}_{n'n} + a^{\mu\mu_1\cdots\mu_D}_\alpha p_\mu \delta_{n'n} \tag{110}$$

From what we had shown before, $(b^\#_\alpha)^{\mu_1\cdots\mu_D}_{n'n}$ are momentum-independent, traceless Hermitian generators of internal symmetry and $a^{\mu\mu_1\cdots\mu_D}_\alpha$ are independent numerical constants. Both these quantities are symmetric in $\mu_1\cdots\mu_D$ because $(B^\#_\alpha)^{\mu_1\cdots\mu_D}_{n'n}$ is, as is evident from (108).

Since A_α cannot take one-particle states off mass-shell we have,

$$[P_\mu P^\mu, A_\alpha] = 0 \tag{111}$$

For D≥1 this implies,

$$[P^{\mu_1}P_{\mu_1},[P^{\mu_2}\cdots[P^{\mu_D},A_\alpha]]\cdots] = 2P_{\mu_1}B^{\mu_1\cdots\mu_D}_\alpha = 0 \tag{112}$$

The last equality follows from (111). So therefore we have,

$$p_{\mu_1}b^{\mu_1\cdots\mu_D}_\alpha(p) = p_{\mu_1}((b^\#_\alpha)^{\mu_1\cdots\mu_D} + a^{\mu\mu_1\cdots\mu_D}_\alpha p_\mu) = 0 \tag{113}$$

the solution is,

$$((b^\#_\alpha)^{\mu_1\cdots\mu_D} = 0 \tag{114}$$

and

$$a^{\mu\mu_1\cdots\mu_D}_\alpha = -a^{\mu_1\mu\cdots\mu_D}_\alpha \tag{115}$$

Now we go over case by case. Lets first take D=0. For this we trivially have $A_\alpha = B_\alpha$, and therefore A_α commutes with P_μ.

For D≥2 we have,

$$\begin{aligned}a^{\mu\mu_1\mu_2\cdots\mu_D}_\alpha &= -a^{\mu_1\mu\mu_2\cdots\mu_D}_\alpha\\ &= -a^{\mu_1\mu_2\mu\cdots\mu_D}_\alpha\\ &= a^{\mu_2\mu_1\mu\cdots\mu_D}_\alpha\\ &= a^{\mu_2\mu\mu_1\cdots\mu_D}_\alpha\\ &= a^{\mu\mu_2\mu_1\cdots\mu_D}_\alpha\\ &= -a^{\mu\mu_1\mu_2\cdots\mu_D}_\alpha\end{aligned} \tag{116}$$

Therefore,

$$a^{\mu\mu_1\mu_2\cdots\mu_D}_\alpha = 0 \tag{117}$$

In (116) we got the antisymmetry in other indices by using (112) with different indices. Thus for D≥2 we have,

$$B_\alpha^{\mu_1\mu_2\cdots\mu_D} = 0 \tag{118}$$

Now lets have a look at the interesting case of D=1. This tells us,

$$a_\alpha^{\mu\mu_1} = -a_\alpha^{\mu_1\mu} \tag{119}$$

For D=1 we have,

$$B_\alpha^\mu = [P^\mu, A_\alpha] = a_\alpha^{\mu\nu} P_\mu \tag{120}$$

where $a^{\mu\nu}$ is antisymmetric in its indices. We see that the commutator of A_α with four-momentum gives a linear combination of four momentum operator. This is reminiscent of ,

$$[P^\mu, J^{\rho\sigma}] = -i\eta^{\nu\rho} P^\sigma + i\eta^{\nu\sigma} P^\rho \tag{121}$$

using (121) we can write,

$$[P^\rho, A_\alpha] = [P^\rho, -\frac{i}{2} a_\alpha^{\mu\nu} J_{\mu\nu} + B_\alpha] = a_\alpha^{\mu\nu} P_\mu \tag{122}$$

This gives us,

$$\boxed{A_\alpha = -\frac{i}{2} a_\alpha^{\mu\nu} J_{\mu\nu} + B_\alpha} \tag{123}$$

So we found that A_α is just the sum of some internal symmetry generator and Lorentz generator.

Over the course of this note we have found, that the generator $\mathcal{G}$ consist only on generators of Poincar group $\mathcal{P}$ and the generators of internal symmetry.

$$\boxed{\mathcal{G} = \mathcal{P} \oplus \mathcal{B}} \tag{124}$$

6 Conclusion and Loopholes

With (124) we have proved that $\mathcal{G}$ is locally isomorphic to the direct product of an internal symmetry group and the Poincar group. Thus the theorem of Coleman and Mandula which we proved in this note demonstrates that the most general Lie algebra of symmetries of the S-matrix contains Momentum generator P_μ , Lorentz generator $J_{\mu\nu}$ and internal symmetry generator B_α. These follow,

$$[P_\mu, B_\alpha] = 0 \qquad [J_{\mu\nu}, B_\alpha] = 0 \tag{125}$$

where B_α has a Lie algebra

$$[B_\alpha, B_\beta] = iC_{\alpha\beta}^\gamma B_\gamma \tag{126}$$

Finally I will list down some loopholes of this no-go theorem.

1. For massless theories ($p^2 = 0$), the argument of antisymmetry of $a_\alpha^{\mu\mu_1\cdots\mu_D}$ does not work. So we can have generators of conformal group which do mix with Poincar group.
2. This theorem does not capture the symmetries of action which do not appear in S-matrix. For example discrete symmetries and spontaneously broken symmetries.
3. Supersymmetry avoids the restriction of Coleman-Mandula theorem by replacing commutators by anti-commutators, effectively enhancing the lie algebra to graded lie algebra [4].

7 References

1. S. Coleman and J. Mandula, “All Possible Symmetries of the S Matrix,” *Phys. Rev.*, vol 159, pp. 1251-1256, Jul 1967

2. S. Weinberg, *The Quantum Theory of Fields, Volume III: Supersymmetry.* Cambridge University Press,2000

3. S. Weinberg, *The Quantum Theory of Fields, Volume II: Modern Applications.* Cambridge University Press,2000

4. R.Haag,J.T. Lopuszanski, and M. Sohnius, “All Possible Generators of Supersymmetries of the S Matrix”, *Nucl. Phys.* B88 (1975) 257.