Supercurrent Superfield for Wess-Zumino Model

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1 Introduction

Our aim is to calculate supercurrent superfield for any arbitrary Kahler Potential and Superpotential(for the Wess Zumino Model). We will be using the Majorana formalism for the calcultion. All the conventions are based on Weinberg ones. Section 26.7 of Weinberg [1] is the primary reference for the following calculations. For alternate formalism we take aid of [2][3].
There are two ways of approaching this problem. Method 1 involves taking the most general Lagrangian for arbitrary Kahler Potential (26.8.6) and then using (26.7.1-2) to calculate the supercurrent and then re-framing it in the superfield language. This method is very tedious. Method 2 is the one we will be using. It involves making use of Conservation law (26.7.26),

$$\gamma^\mu D\Theta_\mu = DX \tag{1}$$

Here X is a Chiral Superfield. We know the formula for X so we can exactly calculate the right hand side. This gives us an idea on what left side should be. In addition we also know the form of Θ_μ for $K = \Phi\Phi^*$ case. So any general supercurrent superfield should reduce to (6) in the $K = \Phi\Phi^*$ case. We will employ these two ideas in finding the supercurrent superfield. All the calculation is done in Majorana Superfied formalism. We will also briefly mention an alternative way to write the superfield which appears a lot in modern literature.
Before we begin let us write down some important formulas that we will be needing. We denote derivatives of K as,

$$K_\Phi = \frac{\partial K}{\partial \Phi} \qquad K^*_\Phi = \frac{\partial K}{\partial \Phi^*} \tag{2}$$

The connection between derivatives of f and K is given as,

$$D_R^2 K_\Phi = -4\frac{\partial f}{\partial \Phi} \qquad D_L^2 K_{\Phi^*} = 4\frac{\partial f^*}{\partial \Phi^*} \tag{3}$$

We would also make heavy use of following commutation(anti) relations,

$$\{D_L, \bar{D}_R\} = -2\partial\!\!\!/ \tag{4}$$

and,

$$[D_R, (\bar{D}_L D_L)] = -4\partial\!\!\!/ D_L \tag{5}$$

Note: We will be supressing indices on Φ. So Φ is actually Φ_i, Whenever there is a $K_{i\bar{j}}$ present the indices should be read as (for example),

$$K_{i\bar{j}}\partial\Phi\partial\Phi^* = K_{i\bar{j}}\partial\Phi_i\partial\Phi^*_j$$

and when there are no K_{ij} the index on Φ is supposed to be repeated, for eg,

$$\Phi\frac{\partial f}{\partial \Phi} = \Phi_i\frac{\partial f}{\partial \Phi_i}$$

2 How to proceed

We already know what the Supercurrent superfield is for the case when $K = \Phi\Phi^*$. It is given by (26.7.21):

$$\Theta_\mu = \frac{i}{12}\left[4\Phi\partial_\mu\Phi - 4\Phi\partial_\mu\Phi^* + ((\bar{D}\Phi^*)\gamma_\mu(D\Phi))\right] \tag{6}$$

This will serve as a guiding light during our calculation. One important thing to observe is that in the general Lagrangian (26.8.6), there is only one term involving K or its derivatives. Hence Supercurrent should contain at max one K or its derivatives in each of its term. This is an important observation because it will aid us in narrowing down the possible choices for the General Supercurrent Superfield. The third term of (6) can be easily generalized for the arbitrary case. Since we can have only one K term in the superfield, the only viable term is:

$$K_{i\bar{j}}(\bar{D}\Phi^*)\gamma_\mu(D\Phi) \tag{7}$$

where,

$$K_{i\bar{j}} = \frac{\partial^2 K}{\partial\Phi\partial\Phi^*} \tag{8}$$

So we have an idea what the third time generalizes to. Now we generalize the first two terms of (6). There are two possible options,

$$\Phi^*\partial_\mu K_{\Phi^*} - \Phi\partial_\mu K_\Phi \tag{9}$$

or,

$$K_{\Phi^*}\partial_\mu\Phi^* - K_\Phi\partial_\mu\Phi \tag{10}$$

A priori we do not know which one should we go with, we will choose latter as a guess and see if it works. Now our superfield will be made from combination of (7) and (10). To see if it is correct, a) the action of $\gamma^\mu D$ on it should give (1) b) It should reduce to (6). In the subsequent sections we will try to give a more compact form to (10).

We would also make use of the following identity a lot,

$$\bar{D}_L D_L D_R K = D_L^2(K_{\Phi^*}D_R\Phi^*) \tag{11}$$

$$= \bar{D}_L(D_L K_{\Phi^*}D_R\Phi^* + K_{\Phi^*}D_L D_R\Phi^*) \tag{12}$$

$$= D_L^2 K_{\Phi^*}D_R\Phi^* - 2D_L K_\Phi^*\bar{D}_L D_R\Phi^* + K_{\Phi^*}D_L^2 D_R\Phi^* \tag{13}$$

By using the commutator(4) the last term goes to zero. We finally get,

$$\bar{D}_L D_L D_R K = 4D_R f^* + 4D_L K_{\Phi^*}\not{\partial}\Phi^* \tag{14}$$

3 Calculation for X

Now we start calculating the right side of,

$$\gamma^\mu D\Theta_\mu = DX \tag{15}$$

where X is a chiral superfield given by,

$$X = \frac{2}{3}Im[\Phi\frac{\partial f}{\partial\Phi} - 3f] \tag{16}$$

This definition has to be modified slighly when the order(power of each term) in Kahler potential is $n > 1$. Then the first term in this expression is divided by n.

$$X = \frac{1}{3i}\left[(\Phi\frac{\partial f}{\partial\Phi} - 3f) - (\Phi\frac{\partial f}{\partial\Phi} - 3f)^*\right] \tag{17}$$

$$X = \frac{1}{3i}\left[\Phi\frac{\partial f}{\partial\Phi} - 3f - \Phi^*\frac{\partial f^*}{\partial\Phi^*} + 3f^*\right] \tag{18}$$

Now using (3) this becomes:

$$X = \frac{1}{3i}\left[\Phi(-\frac{1}{4}D_R^2\frac{\partial K}{\partial\Phi}) - \Phi^*(-\frac{1}{4}D_L^2\frac{\partial K}{\partial\Phi^*}) - 3(f - f^*)\right] \tag{19}$$

$$= \frac{1}{3i}\left[-\frac{1}{4}D_R^2(\Phi\frac{\partial K}{\partial\Phi}) - \frac{1}{4}\Phi^* D_L^2(\frac{\partial K}{\partial\Phi^*}) - 3(f - f^*)\right] \tag{20}$$

Now we act on this with $D = D_L + D_R$,

$$DX = -\frac{i}{12}\left(D_L D_R^2 K + D_R D_L^2 K\right) + i(D_L f - D_R f^*) \tag{21}$$

Now we have calculated the right side of (15) ie the action of D on X. Now we need to find a general Θ_μ which on acted upon by $\gamma^\mu D$ gives (21).

4 Finding Θ_μ

We begin with (8), it can be rewritten as,

$$(\bar{D}\Phi^*)\gamma_\mu K_{i\bar{j}}(D\Phi) = \bar{D}_R\Phi^*\gamma_\mu D_L K_{\Phi^*} \tag{22}$$

Now we act with $\gamma^\mu D_L$. The next few lines are exactly similar to Weinberg Vol 3 (pg 97).

$$\gamma^\mu D_L(\bar{D}\Phi^*)\gamma_\mu K_{i\bar{j}}(D\Phi) = \gamma^\mu D_L\bar{D}_R\Phi^*\gamma_\mu D_L K_{\Phi^*} + 2\bar{D}_R\Phi^* D_L^2 K_{\Phi^*} \tag{23}$$

Using (4) on the first term we get,

$$= 4\partial\!\!\!/\Phi^* D_L K_{\Phi^*} + 2D_R\Phi^* D_L^2 K_{\Phi^*} \tag{24}$$

Finally using (5) on the second term we get (also shifting the bar in second term),

$$= 4\partial\!\!\!/\Phi^* D_L K_{\Phi^*} + 8D_R f^* \tag{25}$$

Now we head to the second set of terms (10),

$$\partial_\mu\Phi K_\Phi - \partial_\mu\Phi^* K_{\Phi^*} \tag{26}$$

Add and subtract the second term,

$$\partial_\mu\Phi K_\Phi + \partial_\mu\Phi^* K_{\Phi^*} - 2\partial_\mu\Phi^* K_{\Phi^*} \tag{27}$$

Thus we get,

$$\partial_\mu K - 2K_{\Phi^*}\partial_\mu\Phi \tag{28}$$

We will begin our calculation with this form of (10),

$$\partial_\mu K - 2\partial_\mu\Phi^* K_{\Phi^*} \tag{29}$$

Applying $\gamma^\mu D_L$ to this gives us,

$$\partial\!\!\!/ D_L K - 2D_L K_{\Phi^*}\partial\!\!\!/\Phi^* \tag{30}$$

$$= -\frac{1}{4}(D_R D_L^2 K - D_L^2 D_R K) - 2D_L K_{\Phi^*}\partial\!\!\!/\Phi^* \tag{31}$$

$$= -\frac{1}{4}D_R D_L^2 + D_R f^* + D_L K_{\Phi^*}\partial\!\!\!/\Phi^* - 2D_L K_{\Phi^*}\partial\!\!\!/\Phi^* \tag{32}$$

$$= -\frac{1}{4} D_R D_L^2 + D_R f^* - D_L K_{\Phi^*} \partial\!\!\!/ \Phi^* \quad (33)$$

So we have,

$$\gamma^\mu D_L(\partial_\mu K - 2\partial_\mu \Phi^* K_{\Phi^*}) = -\frac{1}{4} D_R D_L^2 + D_R f^* - D_L K_{\Phi^*} \partial\!\!\!/ \Phi^* \quad (34)$$

To repeat the same calculation for D_R action, all we need to do is rewrite (26) in terms of $K_\Phi \partial_\mu \Phi$ and rest of the analysis will go through.
Now getting everything together we have from (25) and (34),

$$\Theta_\mu = \frac{i}{12}(4(\partial_\mu \Phi K_\Phi - \partial_\mu \Phi^* K_{\Phi^*}) + K_{i\bar{j}}(\bar{D}\Phi^*)\gamma_\mu(D\Phi)) \quad (35)$$

We choose the coefficient of 4 ,a) from coefficient matching with right hand side of (21) (as shown below) b) so that it reduces to (6).
Using (25) and (34) the action of $\gamma^(\mu) D_L$ gives,

$$\gamma^\mu D_L \Theta_\mu = \frac{i}{12}(-D_R D_L K^2 + 12 D_R f^*) \quad (36)$$

$$= \frac{i}{12} D_R(-D_L K^2 + 12 f^*) \quad (37)$$

We can follow the same procedure for the D_R operator however hermitian adjoint of (37) would do the same job. Thus we get,

$$\gamma^\mu D_R \Theta_\mu = -\frac{i}{12} D_L(-D_L K^2 + 12 f^*)^* \quad (38)$$

$$= \frac{i}{12} D_L(-D_R^2 K + -12 f) \quad (39)$$

Adding (37) and (39)

$$\gamma^\mu D \Theta_\mu = -\frac{i}{12}(D_L D_R^2 K + D_R D_L^2 K) + i(D_L f - D_R f^*) \quad (40)$$

This is exactly the right side that we saw in equation (21). Thus (35) is the most generic superfield.

5 Consistency Check

In the previous sections we stated that the most general superfield is given as,

$$\Theta_\mu = \frac{i}{12}(4(\partial_\mu \Phi K_\Phi - \partial_\mu \Phi^* K_{\Phi^*}) + K_{i\bar{j}}(\bar{D}\Phi^*)\gamma_\mu(D\Phi)) \quad (41)$$

As a consistency check we will see that it reproduces exactly the Superfield given in (26.7.21) of Weinberg for $K = \Phi\Phi^*$. The first two terms simplify readily,

$$\partial_\mu \Phi K_\Phi - \partial_\mu \Phi^* K_{\Phi^*} = \partial_\mu \Phi \Phi^* - \partial_\mu \Phi^* \Phi \quad (42)$$

Since,

$$K_{i\bar{j}} = 1 \quad (43)$$

The last term of (41) becomes

$$K_{i\bar{j}}(\bar{D}\Phi^*)\gamma_\mu(D\Phi)) = (\bar{D}\Phi^*)\gamma_\mu(D\Phi)) \quad (44)$$

Thus we have reproduced all terms of (6). Hence our General Supercurrent Superfield is consistent.

6 Alternate Formalism

So far all the calculation we did were in the convention of Weinberg. However in literature the conventions are slightly different. Literature usually follows Weyl Superfield formalism which we wont get into, however for completeness I will state the result in Weyl Formalism in the next section. In this section I will briefly give an alternative way to write down the superfield. Following [2][3] the supercurrent is usually

$$\Theta = \gamma^\mu \Theta_\mu \tag{45}$$

In this form, the conservation equation is given as,

$$D\Theta = DX \tag{46}$$

Now we can write the first two terms of (41) in a very compact way as,

$$[D_L, \bar{D}_R]K \tag{47}$$

Lets see if it works for the special case of $K = \Phi\Phi^*$

$$[D_L, D_R]\Phi^*\Phi = (D_L D_R - D_R D_L)\Phi\Phi^* \tag{48}$$

$$= D_L(\Phi D_R \Phi^*) - D_R((D_L\Phi)\Phi^*) \tag{49}$$

$$D_L\Phi D_R\Phi^* + \Phi(D_L D_R \Phi^*) - (D_R D_L \Phi)\Phi^* - D_L\Phi D_R \Phi^* \tag{50}$$

first and last term cancel out, we can use the commutator (4) on the middle two terms and we get,

$$\Phi \not{\partial} \Phi^* - \not{\partial} \Phi \Phi^* \tag{51}$$

Thus it works. (47) serves as a good replacement for (26) for the new superfield Θ. The difference between the original superfield and the new definition is that one has been contracted with gamma and hence doesnt contain a Lorenz index. Now we will take up the general case. The action of D_L gives,

$$D_L[D_L, \bar{D}_R] = D_L(D_L\bar{D}_R - \bar{D}_R D_L) \tag{52}$$

$$D_L D_L \bar{D}_R - D_L \bar{D}_R D_L \tag{53}$$

$$D_L D_L \bar{D}_R + 2\not{\partial} D_L + \bar{D}_R D_L D_L \tag{54}$$

$$D_L D_L \bar{D}_R - \frac{1}{2}(D_R D_L^2 + D_L^2 D_R) + \bar{D}_R D_L D_L \tag{55}$$

After shifting the bars we get,

$$-\frac{3}{2}(D_L^2 D_R + D_R D_L^2) \tag{56}$$

We also have,

$$D_L(K_{i\bar{j}}\bar{D}_R\Phi^* D_L\Phi) = -2\not{\partial}\Phi^* D_L K_{\Phi^*} + 4D_R f^* \tag{57}$$

Combining the two bits the superfield can be written as,

$$\Theta = (K_{i\bar{j}}\bar{D}_R\Phi^* D_L\Phi) - \frac{1}{3}[D_L, \bar{D}_R]K \tag{58}$$

The indices of D are repeated here. **We can go between the two superfield formulations by the action of gamma**.

7 Final Results

We will restate the results again in this section. In Weinberg formalism the supercurrent superfield for a general Kahler and Superpotential is given as,

$$\Theta_\mu = \frac{i}{12}(4(\partial_\mu \Phi K_\Phi - \partial_\mu \Phi^* K_{\Phi^*}) + K_{i\bar{j}}(\bar{D}\Phi^*)\gamma_\mu(D\Phi)) \tag{59}$$

In the alternate formalism where the superfield is contracted with gamma we have,

$$\Theta = (K_{i\bar{j}}\bar{D}_R\Phi^* D_L\Phi) - \frac{1}{3}[D_L, \bar{D}_R]K \tag{60}$$

For the sake of completeness we will also state the result in Weyl Superfield formalism [2][3],

$$J_{\alpha\dot{\alpha}} = 2K_{i\bar{j}}(D_\alpha\Phi^i)(\bar{D}_{\dot{\alpha}}\bar{\Phi}^{\bar{i}}) - \frac{2}{3}[D_\alpha, \bar{D}_{\dot{\alpha}}]K \tag{61}$$

8 References

1. S.Weienberg,"Quantum Theory of Fields Vol3:Supersymmetry (chapter 26)",Cambridge Univ Press

2. Z. Komargodski and N. Seiberg,"Comments on Supercurrent Multiplets, Supersymmetric Field Theories and Supergravity", arXiv:hep-th/1002.2228

3. D. Arnold, JP Derendinger and J. Hartong,"On Supercurrent Superfields and Fayet-Iliopoulos Terms in N = 1 Gauge Theories", arXiv:hep-th/1208.1648

4. Ferrara, Zumino,"Transformation properties of Supercurrent", Nucl. Phys. B87

5. N.Seiberg, Talk on Supercurrent given at IAS 2010, https://www.sns.ias.edu/ckfinder/userfiles/files/supercurrents.pdf

9 Another Approach (ignore)

We could have taken the first 2 terms as,

$$\partial_\mu K - 2\Phi^* \partial_\mu K_{\Phi^*} \tag{62}$$

Applying $\gamma^\mu D_L$ to this gives us,

$$\not{\partial} D_L K - 2\Phi^* \not{\partial} D_L K_{\Phi^*} \tag{63}$$

$$= -\frac{1}{4}(D_R D_L^2 K - D_L^2 D_R K) + \frac{1}{2}(\Phi^* D_R D_L^2 K_{\Phi^*} - \Phi^* D_L^2 D_R K_{\Phi^*}) \tag{64}$$

$$= -\frac{1}{4} D_R D_L^2 + D_R f^* - D_L K_{\Phi^*} \not{\partial}\Phi^* + 2\Phi^* D_R \frac{\partial f^*}{\partial \Phi^*} - \frac{1}{2} D_L^2 (\Phi^* D_R K_{\Phi^*}) \tag{65}$$

The last two terms can be simplified as,

$$= -\frac{1}{4} D_R D_L^2 + D_R f^* - D_L K_{\Phi^*} \not{\partial}\Phi^* + 2D_R(\Phi^* \frac{\partial f^*}{\partial \Phi^*}) - 2D_R f^* - \frac{(n-1)}{2} D_L^2 D_R K \tag{66}$$

$$= -\frac{1}{4} D_R D_L^2 + D_R f^* - D_L K_{\Phi^*} \not{\partial}\Phi^* + 2D_R(\Phi^* \frac{\partial f^*}{\partial \Phi^*}) - 2D_R f^* - 2(n-1) D_R f^* + 2(n-1) D_L K_{\Phi^*} \not{\partial}\Phi^* \tag{67}$$

$$= -\frac{1}{4} D_R D_L^2 K + (1-2n) D_R f^* + (2n-3) D_L K_{\Phi^*} \not{\partial}\Phi^* + 2D_R(\Phi^* \frac{\partial f^*}{\partial \Phi^*}) \tag{68}$$